Consistency Conditions on S-Matrix of Spin 1 Massless Particles

Song He

School of Physics, Peking University, Beijing, 100871, China
hesong@pku.edu.cn

Hongbao Zhang

Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada
Department of Astronomy, Beijing Normal University, Beijing, 100875, China
hzhang@perimeterinstitute.ca

ABSTRACT: Motivated by new techniques in the computation of scattering amplitudes of massless particles in four dimensions, like BCFW recursion relations, the question that how much structure of the S-matrix can be determined from purely S-matrix arguments has received new attention. The BCFW recursion relations for massless particles of spin 1 and 2 imply that the whole tree-level S-matrix can be determined in terms of three-particle amplitudes evaluated at complex momenta. However, the known proofs of the validity of the relations rely on the Lagrangian of the theory, either by using Feynman diagrams explicitly or by studying the effective theory at large complex momenta. This means that a purely S-matrix proof of the relations is still missing. The aim of this paper is to provide such a proof for spin 1 particles by extending the four-particle test introduced by Benincasa and Cachazo in [1] to arbitrary numbers of particles. We show how n-particle tests imply that the rational function built from the BCFW recursion relations possesses all the correct factorization channels including holomorphic and anti-holomorphic collinear limits, which thus produces the correct S-matrix of the theory.

Contents

1. Introduction

As an alternative to Feynman diagram analysis, BCFW construction is a powerful tool for constructing tree-level amplitudes in terms of sub-amplitudes with fewer external particles [2, 3]. The class of theories whose amplitudes can be completely determined by such a BCFW construction from lower amplitudes are called constructible [1], and have been proven to include Yang-Mills theory, General Relativity and more general two derivative theories such as QCD, $\mathcal{N} = 4$ SYM theory and $\mathcal{N} = 8$ Supergravity [3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. However, all of these proofs rely heavily on Lagrangian of the theory, either by using Feynman diagrams explicitly [3, 4, 5, 6, 7], or by studying the effective theory at large complex momenta [8, 9, 10].

The purpose of this paper is to address the constructibility of theories of spin 1 massless particles from a complementary perspective. Instead of assuming a priori knowledge of Yang-Mills Lagrangian with its Feynman diagrams, we here directly construct candidates for the tree-level amplitudes via BCFW recursion relations and then prove such a construction is consistent and the resultant amplitudes are indeed the correct physical amplitudes, given that some conditions on three-particle couplings are satisfied.

Speaking specifically, Benincasa and Cachazo have discussed the consistency conditions on four particle amplitudes constructed from three particle ones for massless particles with general spins [1]. In particular, as for spin 1 massless particles, such consistency conditions require that the negative dimension couplings should be absent and the dimensionless coupling constants should be the structure constants of a Lie group. However, a generalization to amplitudes with more external particles is still lacking and the aim of this paper is to fill this gap. It turns out that no further consistency conditions are needed for higher-point amplitudes and BCFW construction automatically gives the correct physical amplitudes with five or more particles. Therefore, along with the result for four-particle test in [1], the present paper provides the first purely S-matrix proof of constructibility of theories for spin 1 massless particles[1].

The paper is organized as following. After a brief review of scattering amplitudes for massless particles and its construction via BCFW recursion relations in Section 2, we set out in Section 3 to prove the consistency conditions on scattering amplitudes of spin 1 massless particles by induction. Conclusion and discussions are presented in the end.

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[1]The similar result has also been obtained independently by Schuster and Toro [13].

2. Preliminaries

2.1 S-matrix of massless particles

To set our notation and introduce convenient spinor language, we review S-matrix first for general theories in four dimensional Minkowski spacetime and then for theories with massless particles.

Probability for scattering process from asymptotic initial state to final state is of particular physical interests, and it can be calculated from physical inner-product of multi-particle states, i.e.,

$$_{out}\langle p_1, p_2, ..., p_m | p_1', p_2', ..., p_{n-m}' \rangle_{in} = \langle p_1, p_2, ..., p_m | S | p_1', p_2', ..., p_{n-m}' \rangle, \qquad (2.1)$$

where $S = I + iT$ is an unitary operator. Then we can define the scattering amplitude M by

$$\langle p_1, p_2, ..., p_m | iT | p_1', p_2', ..., p_{n-m}' \rangle = \delta^4(\sum_{i=1}^{m} p_i - \sum_{i=1}^{n-m} p_i') M(\{p_1', ..., p_{n-m}'\} \to \{p_1, ..., p_m\}). \qquad (2.2)$$

Instead of working with both ingoing and outgoing particles, we can define scattering amplitude with only outgoing particles by using $p_{m+1} = -p_1', p_{m+2} = -p_2', ..., p_n = -p_{n-m}'$. As a result, the probability of any process which involves n particles in total can be calculated by analytically continuing $M_n(p_1, p_2..., p_n)$.

For any Poincare-invariant theory of massless particles in four dimensional Minkowski spacetime, one-particle states, from which multi-particle states are constructed, are irreducible massless representations of Poincare group. An irreducible massless representation is labeled by the on-shell momentum p satisfying $p^2 = 0$ and helicity $h = \pm s$ where s is the spin of the particle. Furthermore, any on-shell momentum of massless particle can be decomposed into

$$p^\mu = \lambda^a (\sigma^\mu)_{a\dot{a}} \tilde{\lambda}^{\dot{a}} \qquad (2.3)$$

with $\sigma^\mu = (1, \sigma^i)$. Of course this decomposition is not unique, but only up to a little group transformation $\lambda \to t\lambda, \tilde{\lambda} \to t^{-1}\tilde{\lambda}$. In addition, it is noteworthy that for complex momentum, which is essential for BCFW construction and naturally defined by using complexified Lorentz group $SL(2, C) \times SL(2, C)$, λ and $\tilde{\lambda}$ are completely independent. Note that any Lorentz invariants can be constructed from basic invariants, i.e.,

$$\langle \lambda, \lambda' \rangle = \varepsilon^{ab} \lambda_a \lambda_b', [\tilde{\lambda}, \tilde{\lambda}'] = \varepsilon^{\dot{a}\dot{b}} \tilde{\lambda}_{\dot{a}} \tilde{\lambda}_{\dot{b}}'. \qquad (2.4)$$

An important example is the invariant from two on-shell momenta, $p^\mu = \lambda \sigma^\mu \tilde{\lambda}$ and $q^\mu = \lambda' \sigma^\mu \tilde{\lambda}'$, i.e.,

$$(p+q)^2 = 2p \cdot q = \langle \lambda, \lambda' \rangle [\tilde{\lambda}, \tilde{\lambda}']. \tag{2.5}$$

All information of massless particles is encoded in pairs of spinors and their helicities, from which an amplitude can be constructed. This can be clearly seen from Feynman diagrams, where all components, including propagators, vertices and polarization vectors are functions of spinors and helicities. In particular, since an n particle amplitude M_n is Lorentz invariant, we conclude that it is a function of n helicities and basic Lorentz invariants constructed from n pairs of spinors, $\langle i, j \rangle \equiv \langle \lambda_i, \lambda_j \rangle$ and $[i, j] = [\tilde{\lambda}_i, \tilde{\lambda}_j]$ for $1 \le i, j \le n$. In addition, for tree-level amplitude, such a function is rational.

2.2 BCFW construction

The BCFW recursion relations can be schematically written as

$$M_n^{(l,m)}(\{p_i, h_i, a_i\} | i = 1, 2, ..., n) =$$
$$\sum_{I,h,a} M_{|I|+1}(I(z_I), \{-P_I(z_I), -h, a\}) \frac{1}{P_I^2} M_{|\bar{I}|+1}(\{P_I(z_I), h, a\}, \bar{I}(z_I)). \tag{2.6}$$

Here some explanations are needed. We have picked two reference particles (l, m) with their momenta in sub-amplitudes deformed as[2],

$$\lambda^{(l)}(z) = \lambda^{(l)} + z\lambda^{(m)}, \tilde{\lambda}^{(m)}(z) = \tilde{\lambda}^{(m)} - z\tilde{\lambda}^{(l)}. \tag{2.7}$$

where for each subset $I \subseteq \{1, 2, ..., n\}$ with $l \in I$ and $m \in \bar{I}$, the parameter z is valued at the pole of the amplitude, z_I, corresponding to sending the momentum of internal legs $P_I(z_I) = \sum_{i \in I} p_i(z_I) = -\sum_{i \in \bar{I}} p_i(z_I)$ on shell, with $p_i(z_I) = p_i$ undeformed for $i \ne l, m$. The summation is over all divisions of external particles into I and $\bar{I}$ with $l \in I$ and $m \in \bar{I}$, as well as the helicity h and color a of internal leg. Every term in the summation is a product of an $|I| + 1$ particle sub-amplitude and an $|\bar{I}| + 1$ particle sub-amplitude where $|I|$ and $|\bar{I}|$ are numbers of external particles in the subset I and $\bar{I}$, and there is one on-shell internal leg with $\{\mp P_I(z_I), \mp h, a\}$ for each of the sub-amplitude, respectively. In addition, $P_I = \sum_{i \in I} p_i = -\sum_{i \in \bar{I}} p_i$ is the off-shell momentum of the propagator without deformation.

Details of the proof of BCFW recursion relations for amplitudes in gauge theories and gravity, as well as more general theories can be found in [3, 4, 5, 6, 7, 8, 9, 10]. Here

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[2] The order of reference particles are relevant, i.e., (l, m) and (m, l) correspond to different deformations.

we shall not repeat the proof but only give some explanations. After the deformation, the tree-level amplitude becomes a rational function of z and the key point for the proof is to show it vanishes as z goes to infinity, i.e., $\lim_{z \to \infty} M_n^{(l,m)}(z) = 0$. Given this remarkable property, we have $\oint_C M_n^{(l,m)}(z)/z = 0$ when the contour C encloses all poles of the function, and the residue theorem implies

$$M_n^{(l,m)} \equiv M_n^{(l,m)}(z=0) = -\sum_{z_I} res_{z=z_I} \frac{M_n^{l,m}(z)}{z}, \qquad (2.8)$$

which essentially gives Eq.(2.6).

As pointed out before, in this paper we assume no a prior knowledge on how to determine M_n by the Yang-Mills Lagrangian with Feynman diagrams and then check if it satisfies Eq.(2.6). Alternatively, we consider Eq.(2.6) as our starting point to *define* a rational function $M_n^{(l,m)}$, which serves as a candidate of the amplitude, and then prove the construction is consistent and the rational function obtained is indeed the correct amplitude M_n. An important remark is that the construction only works for deformations on any two particles with helicities $(+,+)$, $(+,-)$, and $(-,-)$, which we shall name as good deformations, but not for the case $(-,+)$, which is called bad deformation. This has been proved using Yang-Mills Lagrangian and its Feynman diagrams [2, 3]. We shall also verify it below in our purely S-matrix proof.

3. Consistency conditions on tree-level amplitudes of spin 1 massless particles

In this section, we shall determine consistency conditions for any tree-level amplitude to be constructed from sub-amplitudes with fewer external particles using BCFW construction. It turns out that non-trivial constraints only come from consistency conditions on four particle amplitude constructed from three particle ones, or the four-particle test [1], which will be briefly summarized in 3.1. Once the test is passed, the correct physical n particle amplitude can be consistently constructed from lower amplitudes, which we shall prove in 3.3 and 3.4.

We prove this by induction. Suppose the consistency conditions are satisfied for amplitudes with 4,...,$n-1$ particles with $n \geq 5$. There are two different versions, a weak version and a strong version, of these conditions. The strong version means that $M_k^{(i,j)} = M_k^{(l,m)}$ for $k = 4, ..., n-1$ and any $1 \leq i, j, l, m \leq k$ as long as all deformations are good, and the amplitude constructed this way has all correct factorization channels, yielding the correct physical amplitudes. The weak version only states that $M_k(1, 2, ..., k)$ can be constructed by lower amplitudes using some deformations which give the same result, and this is enough to ensure it to be the correct physical amplitude.

Here we only prove the weak version. Suppose we only have $M_k^{(i,i-1)} = M_k^{(i,i+1)}$ and $M_k^{(i-1,i)} = M_k^{(i+1,i)}$ for $1 \le i \le k$, as long as all deformations involved are good, and the amplitude constructed this way is the correct physical amplitude for $k = 4, ..., n-1$. Then we shall prove that the weak version of consistency conditions are also satisfied for n particle amplitudes, which is enough for our purpose.

In 3.3 we shall prove $M_n^{(i,i-1)} = M_n^{(i,i+1)}$ and $M_n^{(i-1,i)} = M_n^{(i+1,i)}$ for $1 \le i \le n$, as long as all deformations involved are good. The proof that these equalities have guaranteed it to possess correct factorization channels, including holomorphic and anti-holomorphic collinear limits, will be presented in 3.4.

3.1 Three particle amplitudes and four-particle test

The starting point of [1] is that three particle amplitudes $M_3(\{p_i, h_i, a_i\}|i = 1, 2, 3)$ for spin s massless particles in four dimensional Minkowski spacetime are completely determined by Poincare symmetry, up to coupling constants. They are either holomorphic or anti-holomorphic[3], i.e.,

$$M_3(\{p_i, h_i, a_i\}|i = 1, 2, 3) = \kappa_{a_1 a_2 a_3} \langle 1, 2 \rangle^{d_3} \langle 2, 3 \rangle^{d_1} \langle 3, 1 \rangle^{d_2}, \qquad (3.1)$$

for $h_1 + h_2 + h_3 < 0$, and

$$M_3(\{p_i, h_i, a_i\}|i = 1, 2, 3) = \kappa'_{a_1 a_2 a_3} [1, 2]^{-d_3} [2, 3]^{-d_1} [3, 1]^{-d_2} \qquad (3.2)$$

for $h_1 + h_2 + h_3 > 0$. Here $d_1 = h_1 - h_2 - h_3$, $d_2 = h_2 - h_3 - h_1$, and $d_3 = h_3 - h_1 - h_2$. In addition, $\kappa_{a_1 a_2 a_3}$ and $\kappa'_{a_1 a_2 a_3}$ are coupling constants for particles with colors a_1, a_2 and a_3, which can be separated into dimensionless coupling constants $f_{a_1 a_2 a_3}$, and generically dimensionful coupling constants κ and κ', which are independent of color indices but can have helicity dependence. In fact, a simple dimension analysis shows that both κ and κ' have the dimension $1 - |h_1 + h_2 + h_3|$ for the case of spin s, which equals $1 - 3s$ for $+++$ and $---$ couplings and $1 - s$ for other cases. For $s = 1$ we denote them as $\kappa^{[-2]}(\kappa'^{[-2]})$ and $\kappa^{[0]}(\kappa'^{[0]})$, respectively. A basic observation on dimensionless coupling constants is that for odd s, $f_{a_1 a_2 a_3}$ are antisymmetric with respect to any two subscripts since in this case d_1, d_2 and d_3 are all odd.

Next one can build the four particle tree-amplitudes from three particle ones by means of BCFW recursion relations. However, as shown in [1], one needs consistency condition on the amplitude, i.e., four particle test: different constructions by deforming

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[3]The fact that these amplitudes are either holomorphic or anti-holomorphic is simply due to the physical condition that any three particle amplitude vanishes for real momenta. For instance, when one takes the limit of real momenta, if $h_1 + h_2 + h_3 < 0$, all anti-holomorphic coupling constants $\kappa'_{a_1 a_2 a_3}$ must vanish to avoid a possible divergence.

particles $(1, 2)$ and $(1, 4)$ must give the same result $M_4^{(1,2)} = M_4^{(1,4)}$. This simple condition imposes severe constraints on non-trivial theories with non-zero coupling constants, which can be summarized as[4]: (1) $\kappa^{[-2]} = \kappa'^{[-2]} = 0$, i.e., there is no $+ + +$ or $- - -$ coupling, (2) the dimensionless coupling constants must conform to Jacobi condition, i.e., $\sum_e (f_{ade} f_{ebc} + f_{ace} f_{edb} + f_{abe} f_{ecd}) = 0$.

As will become clear, the first constraint is crucial for our proof in 3.3. In [1], this has also been shown to come from the condition for constructibility by analysis of Lagrangian and Feynman diagrams. From the Lagrangian point of view, this excludes higher derivative terms like $(F^2)^2$. Here we want to stress that this constraint is part of the consistency conditions on amplitudes constructed by BCFW recursion relations, which holds without the assumption of Lagrangian and Feynman diagrams.

Let us focus on the second constraint, which, together with the fact that each f_{abc} is totally antisymmetric, implies that f_{abc} constitute the structure constants of a Lie algebra. We shall assume in the following that the Lie algebra is $su(\mathfrak{n})$[5], which is our main interest. Suppose T^a to be the generators of $su(\mathfrak{n})$, which satisfy $[T^a, T^b] = f_{abc} T^c$. Since we have assumed consistency conditions on $M_k(1, 2, ..., k)$ for $k = 3, ..., n - 1$, which guarantee them to be the correct physical amplitudes, it is well known, at least for $su(\mathfrak{n})$, that we can do the color decomposition for any tree amplitudes,

$$M_k(\{p_i, h_i, a_i\}|i = 1, ..., k) = \sum_{\sigma \in S_k/C_k} Tr(T^{a_{\sigma(1)}}...T^{a_{\sigma(k)}}) M_k^P(\sigma(1^{h_1}), ..., \sigma(k^{h_k})). \quad (3.3)$$

for $k = 3, ..., n - 1$. Here S_k is the permutation group and C_k is the corresponding cyclic subgroup. In addition, $M_k^P(1^{h_1}, ..., k^{h_k})$, with i^{h_i} referring to $\{p_i, h_i\}$, are called the color-ordered amplitudes, or partial amplitudes. We want to show that the same decomposition can also be done for tree-level n particle amplitudes constructed by recursion relations for $n \geq 4$, using any good deformation, i.e.,

$$M_n^{(l,m)}(\{p_i, h_i, a_i\}|i = 1, ..., n) = \sum_{\sigma \in S_n/C_n} Tr(T^{a_{\sigma(1)}}...T^{a_{\sigma(n)}}) M_n^{P(l,m)}(\sigma(1^{h_1}), ..., \sigma(n^{h_n})).$$
$$(3.4)$$

The key point to justify Eq.(3.4) is the identity for $su(\mathfrak{n})$

$$\sum_{a_I} Tr(T^{a_1}...T^{a_i} T^{a_I}) Tr(T^{a_I} T^{a_{i+1}}...T^{a_n}) = Tr(T^{a_1}...T^{a_n}). \quad (3.5)$$

Since any lower amplitudes, from which the L.H.S. of Eq.(3.4) is constructed, can be decomposed as in Eq.(3.3), then by plugging Eq.(3.3) into Eq.(2.6) and using Eq.(3.5)

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[4]We do not repeat the proof here, and details can be found in [1].

[5]Note that this $\mathfrak{n}$ has nothing to do with n, the number of external particles in amplitudes.

to carry out the summation over a_I, we arrive at the R.H.S. of Eq.(3.4), where $M^{P^{(l,m)}}_n$ is constructed from lower partial amplitudes and eventually M_3^P. In addition, combining the $k = 3$ case of Eq.(3.3) with Eq.(3.1) or Eq.(3.2) shows that M_3^P is just the R.H.S. of Eq.(3.1) with coupling constants $\kappa_{a_1a_2a_3}$ replaced by κ, or the R.H.S. of Eq.(3.2) with $\kappa'_{a_1a_2a_3}$ replaced by κ'.

These partial amplitudes actually contain all the kinematic information, and will be the major objects for study in the following. So we henceforth omit the superscript P. Such partial amplitudes are much simpler than the full ones due to several reasons. First of all, they are cyclic-symmetric for its n external legs, i.e.,

$$M_n(p_1, p_2, ..., p_n) = M_n(p_{\sigma(1)}, p_{\sigma(2)}, ..., p_{\sigma(n)}). \tag{3.6}$$

for any $\sigma \in C_n$.

More importantly, they only receive contributions from diagrams with a certain cyclic ordering of the external legs. An important observation is that all poles of these partial amplitudes merely come from those channels with adjacent momenta, like $s_{i,...,j} = (p_i + p_{i+1}... + p_{j-1} + p_j)^2$. This thus vastly reduced the number of terms that can appear in their BCFW construction. For example, if we want to use the recursion relations by deforming two adjacent particles of an n particle partial amplitude, say $(1, n)$, all divisions of the set $\{1, 2, ..., n\}$ we need to consider are only those of the form $I = \{1, 2, ...i\}$ and $\bar{I} = \{i + 1, ..., n\}$, with $n - 3$ terms, i.e., $2 \leq i \leq n - 2$, instead of any subsets I with $1 \in I$ and $n \in \bar{I}$, which is the case for full amplitudes. The result can be schematically summarized as

$$M_n^{(n,1)}(p_i, h_i | i = 1, 2, ..., n) =$$
$$\sum_{i=2}^{n-2} \sum_{h=\pm} M_{i+1}(1(z_i), ..., i, \{-P_i(z_i), -h\}) \frac{1}{P_i^2} M_{n-i+1}(\{P_i(z_i), h\}, i + 1, ..., n(z_i)).$$
$$\tag{3.7}$$

3.2 A note on notations

Before proceeding to prove our consistency conditions, we would like to pause a moment to fix our notations, which will make our formula compact.

Following [1], we denote a pair of spinors $\{\lambda^{(i)}, \tilde{\lambda}^{(i)}\}$ corresponding to an on-shell momentum p_i by i for $i = 1, ..., n$. Now in a deformation on (i, j), we use a Greek letter as superscript of i to denote the left-handed spinor of i being shifted, while the same letter as subscript of j is used to denote the right-handed one of j being shifted. Deformations on different pairs of particles will be represented by different Greek letters. For example, as illustrated in Figure 1, 2, 3 and 4, the deformation on $(1, 2)$ results in 1^α and 2_α, while the one on $(1, n)$ yields 1^β and n_β.

Furthermore, in both sub-amplitudes of a factorization, momenta are understood to be deformed with the parameter z at the pole of the original amplitude, which keeps momenta of internal legs in this factorization on-shell, as required by the recursion relations. Therefore different factorizations from deforming the same pair of particles have different parameters of deformations, and to label momenta in these factorizations, we need to add subscripts representing different factorizations to the same Greek letter, which are shown in Figure 1,2, 3 and 4, where α and β are short for α_{n-1} and β_{n-1}.

For those on-shell internal legs, we use $i^\alpha \oplus ... \oplus j$ to represent a pair of spinors whose momentum is given by $P = p_{i^\alpha} + ... + p_j$ (up to little group transformation), while the pair of spinors representing $P = -(p_{i^\alpha} + ... + p_j)$ is denoted by $-(i^\alpha \oplus ... \oplus j)$. Momentum conservation ensures that we can use either $i^\alpha \oplus ... \oplus j$ or $-(k_\alpha \oplus ... \oplus l)$ for an internal leg from deformation on (i, k), where $i, ..., j$ are all other particles in the same sub-amplitude, and $k, ..., l$ are all particles except the internal leg in the sub-amplitude on the other side of the propagator. We also explicitly use $\pm h$ to represent opposite helicities of internal legs on two sides of the propagator. Finally, $(p_i + ... + p_j)^2$ is denoted by $|i \oplus ... \oplus j|^2$ in the propagator.

3.3 Proof of $M_n^{(i,i-1)} = M_n^{(i,i+1)}$ and $M_n^{(i-1,i)} = M_n^{(i+1,i)}$

The first step is to prove $M_n^{(1^\alpha,2_\alpha)} = M_n^{(1^\beta,n_\beta)}$. According to the BCFW construction, the n particle partial amplitude with particle 1 and 2 deformed can be constructed as,

$$
\begin{aligned}
M_n^{(1^\alpha,2_\alpha)} = &\sum_{h=\pm} M_3(n, 1^{\alpha_{n-1}}, \{-(n \oplus 1^{\alpha_{n-1}}), -h\}) \frac{1}{|n \oplus 1|^2} \\
&\times M_{n-1}(\{n \oplus 1^{\alpha_{n-1}}, h\}, 2_{\alpha_{n-1}}, 3, ..., n-1) \\
&+ \sum_{i=3}^{n-2} \sum_{h=\pm} [M_{n-i+2}(i+1, ..., n, 1^{\alpha_i}, \{2_{\alpha_i} \oplus ... \oplus i, h\}) \frac{1}{|2 \oplus ... \oplus i|^2} \\
&\times M_i(\{-(2_{\alpha_i} \oplus ... \oplus i), -h\}, 2_{\alpha_i}, ..., i)].
\end{aligned}
\tag{3.8}
$$

Here we have divided the sum over different ways of factorizations into two parts, the former of which corresponding to $i = n-1$ is denoted by A while the latter, the sum over i from 3 to $n-2$, is denoted as B. These two terms are shown in the first lines of figure 1 and figure 2, respectively. As mentioned before, we add i as subscripts to α since parameters $z(\alpha_i)$ of deformations are different for different factorizations.

Similarly, the amplitude with particle 1 and n deformed can also be constructed, i.e.,

$$
M_n^{(1^\beta,n_\beta)} = \sum_{h=\pm} M_{n-1}(3, ..., n-1, n_{\beta_{n-1}}, \{1^{\beta_{n-1}} \oplus 2, h\}) \frac{1}{|1 \oplus 2|^2}
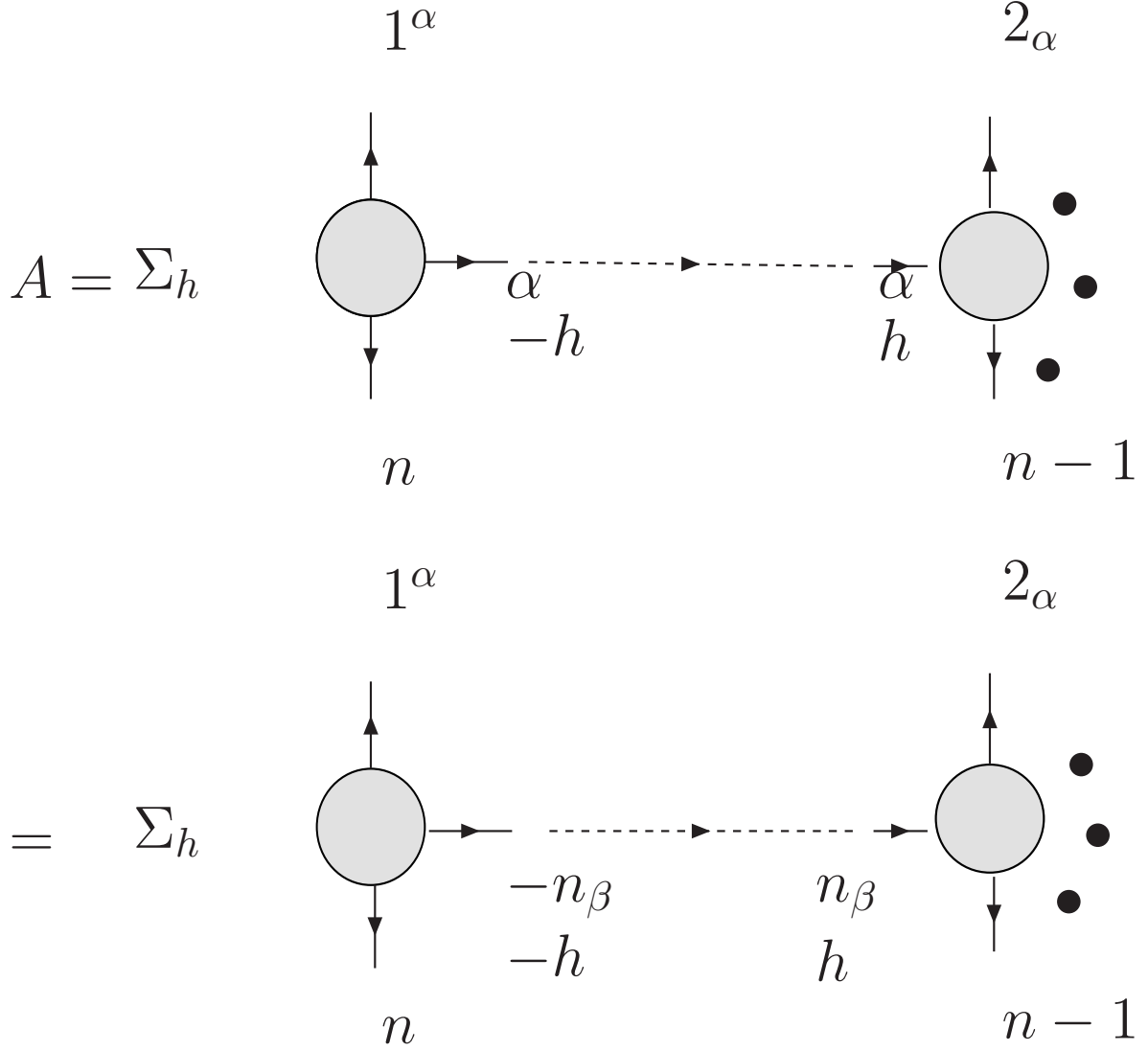$$

Figure 1: Terms in $M_n^{(1,2)}$ with particle 1 in three amplitudes, where dots denote other external particles and dashed lines are off-shell propagators, α and β are short for α_{n-1} and β_{n-1}. In the second line, we use $n_\beta = n \oplus 1^\alpha$ for internal legs.

$$\times M_3(\{-(1^{\beta_{n-1}} \oplus 2), -h\}, 1^{\beta_{n-1}}, 2)$$
$$+ \sum_{j=3}^{n-2} \sum_{h'=\pm} [M_{n-j+1}(j+1, ..., n_{\beta_j}, \{-((j+1) \oplus ... \oplus n_{\beta_j}), h'\}) \frac{1}{|1 \oplus ... \oplus j|^2}$$
$$\times M_{j+1}(\{(j+1) \oplus ... \oplus n_{\beta_j}, -h'\}, 1^{\beta_j}, 2, ..., j)], \tag{3.9}$$

where the term in the first two lines with $j = 2$ is denoted as A' and the rest, the sum over j from 3 to $n-2$, is denoted by B'. These are shown in first lines of figure 3 and
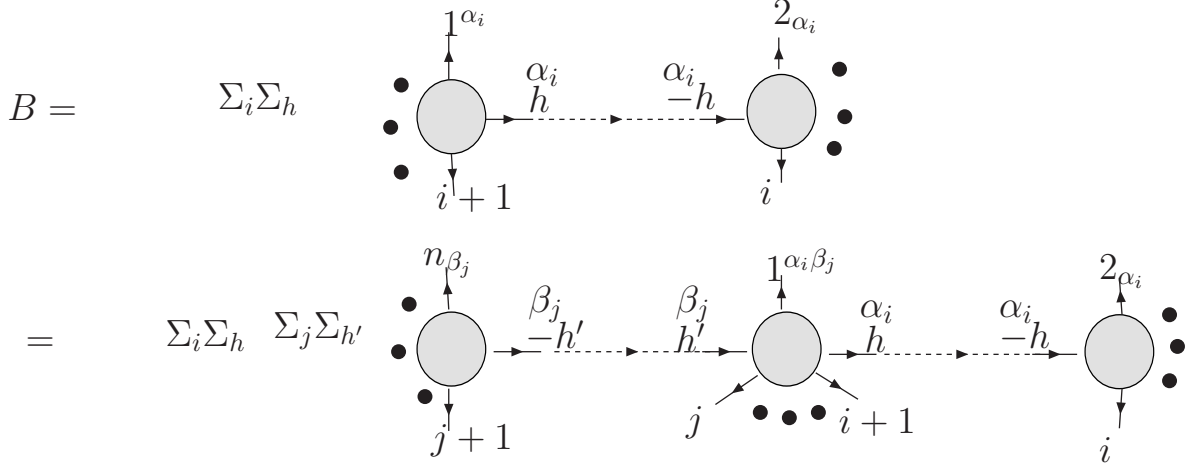
Figure 2: Other terms in $M_n^{(1,2)}$ where dots denote other external particles and dashed lines are off-shell propagators. In the second line we further factorize the left amplitude by deforming the pair $(1, n)$.

figure 4, respectively.

To proceed, first we notice that by deformation on $(1^{\alpha_i}, n)$, as illustrated in the second line of figure 2, B can be factorized further as

$$B = \sum_{i=3}^{n-2} \sum_{j=i}^{n-2} \sum_{h=\pm} \sum_{h'=\pm} [M_{n-j+1}(j+1, ..., n_{\beta_j}, \{-((j+1) \oplus ... \oplus n_{\beta_j}), -h'\})$$

$$\times \frac{1}{|(j+1) \oplus ... \oplus n|^2} M_{j-i+3}(\{(j+1) \oplus ... \oplus n_{\beta_j}, h'\}, 1^{\alpha_i \beta_j}, \{2_{\alpha_i} \oplus ... \oplus i, h\}, i+1, ..., j)$$

$$\times \frac{1}{|2 \oplus ... \oplus i|^2} M_i(\{-(2_{\alpha_i} \oplus ... \oplus i), -h\}, 2_{\alpha_i}, ..., i)]. \tag{3.10}$$

Here in the summation over j from i to $n-2$, when $j=i$, the sub-amplitude M_{j-i+3} is just a three particle amplitude and external legs $\{i+1, ..., j=i\}$ in it are understood to be an empty set in this case.

An important observation is that here we can use β_j with $j = i, ..., n-2$ for these deformations on $(1^{\alpha_i}, n)$ just as those deformations on $(1, n)$, which can be justified as following. Suppose we denote these deformations with super(sub)scripts μ_j. Note that $\tilde{\lambda}^{(1^{\alpha_i})} = \tilde{\lambda}^{(1)}$ and $\lambda^{(n_{\mu_j})} = \lambda^{(n)}$. Then $\tilde{\lambda}^{(n_{\mu_j})} = \tilde{\lambda}^{(n)} - z(\mu_j) \tilde{\lambda}^{(1)}$ where the parameter $z(\mu_j)$ for a factorization j is determined by the on-shell condition of $(j+1) \oplus ... \oplus n_{\mu_j}$ which gives exactly the same equation for $z(\mu_j)$ as that of $z(\beta_j)$. So we have $z(\mu_j) = z(\beta_j)$, which further implies $\tilde{\lambda}^{(n_{\mu_j})} = \tilde{\lambda}^{(n_{\beta_j})}$ and $\lambda^{(1^{\alpha_i \mu_j})} = \lambda^{(1^{\alpha_i})} + z(\beta_j) \lambda^{(n)} = \lambda^{(1^{\alpha_i \beta_j})}$, where $1^{\alpha_i \beta_j}$ is understood as the composition of two deformations on particle 1, with the left one done first.
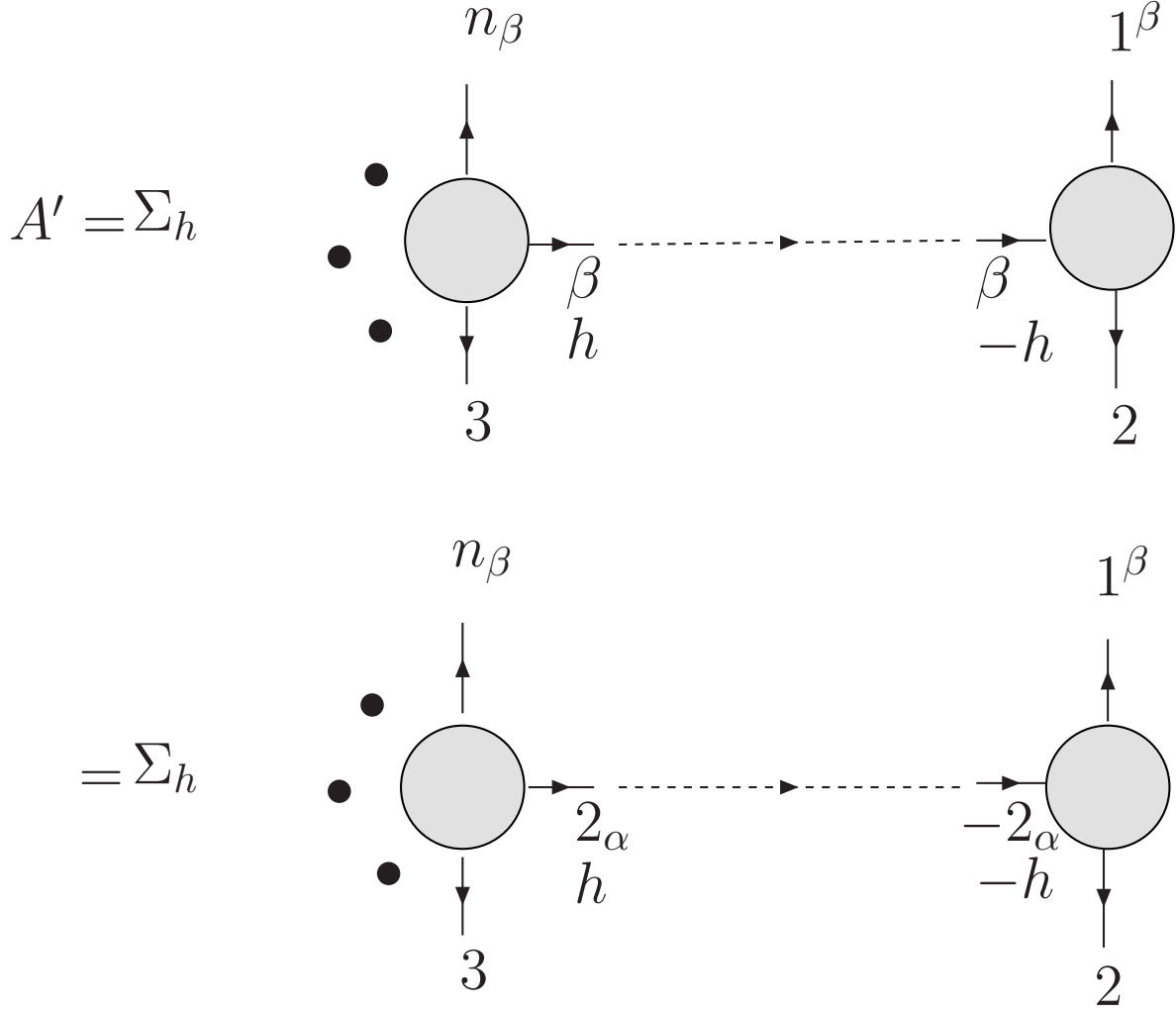
Figure 3: Terms in $M_n^{(1,n)}$ with particle 1 in three amplitudes, where dots denote other external particles and dashed lines are off-shell propagators, α and β are short for α_{n-1} and β_{n-1}. In the second line we use $2_\alpha = 1^\beta \oplus 2$ for internal legs, then the left amplitude is the same as that in A.

Similarly, as shown in the second line of figure 4, B' can be factorized by deforming $(1^{\beta_j}, 2)$,

$$B' = \sum_{j=3}^{n-2}\sum_{i=3}^{j}\sum_{h=\pm}\sum_{h'=\pm}[M_{n-j+1}(j+1,...,n_{\beta_j},\{-((j+1)\oplus...\oplus n_{\beta_j}),-h'\})$$

$$\times\frac{1}{|(j+1)\oplus...\oplus n|^2}M_{j-i+3}(\{(j+1)\oplus...\oplus n_{\beta_j},h'\},1^{\beta_j\,\alpha_i},\{2_{\alpha_i}\oplus...\oplus i,h\},i+1,...,j)$$
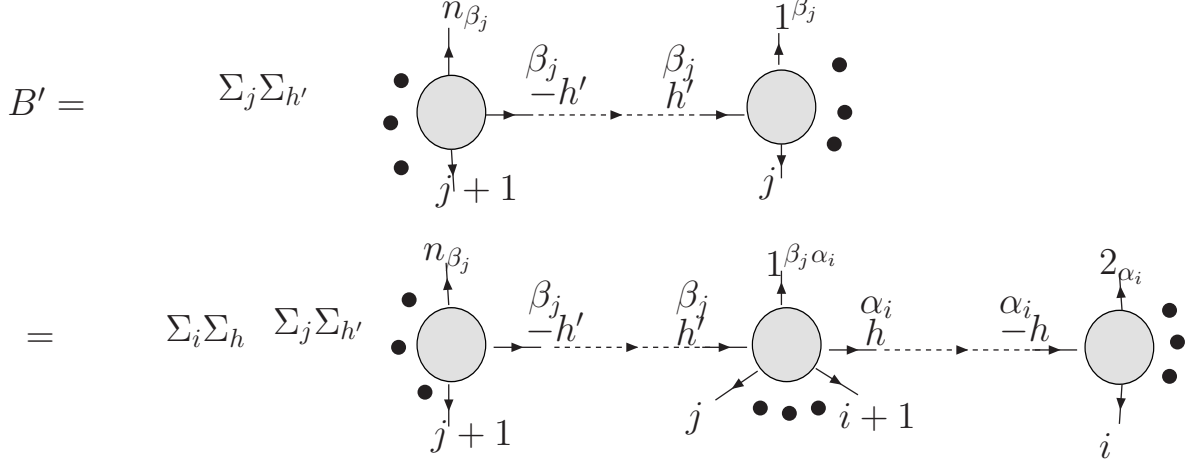
Figure 4: Other terms in $M_n^{(1,n)}$ where dots denote other external particles and dashed lines are off-shell propagators. In the second line we further factorize the right amplitude by deforming the pair $(1, n)$, then the second line is the same as that in B.

$$\times \frac{1}{|2 \oplus ... \oplus i|^2} M_i(\{-(2_{\alpha_i} \oplus ... \oplus i), -h\}, 2_{\alpha_i}, ..., i)]. \tag{3.11}$$

Here we justify the use of α_i for the same reason as before. In $1^{\beta_j \alpha_i}$, the order of actions of two deformations on particle 1 are reversed from that in $1^{\alpha_i \beta_j}$. In both cases, the right-handed spinor $\tilde{\lambda}^{(1)}$ remains unchanged, while $\lambda^{(1^{\beta_j \alpha_i})} = \lambda^{(1)} + z(\beta_j)\lambda^{(n)} + z(\alpha_i)\lambda^{(2)} = \lambda^{(1^{\alpha_i \beta_j})}$. So we have $1^{\beta_j \alpha_i} = 1^{\alpha_i \beta_j}$. In addition, note that the summation over i and j in both Eq.(3.10) and Eq.(3.11) is actually the summation over all i, j with $3 \le i \le j \le n-2$. Therefore, as shown in figures 2 and 4, we conclude that $B = B'$.

Now we proceed to prove $A = A'$. Since in A and A' we only encounter α_{n-1} and β_{n-1}, we henceforth omit the subscript $n-1$ and use α and β, just as we did in figures1 and 3. By Eq.(1),Eq.(4),Eq.(3), and Eq.(6) in the Appendix, we obtain $n \oplus 1^\alpha = n_\beta$ and $1^\beta \oplus 2 = 2_\alpha$. Taking into account $|i \oplus j|^2 = \langle i, j \rangle [i, j]$, we have the following simplified expressions for A and A',

$$A = \sum_{h < h_n + h_1} M_3(n^{h_n}, (1^\alpha)^{h_1}, (-n_\beta)^{-h}) \frac{1}{\langle n, 1 \rangle [n, 1]} M_{n-1}(n_\beta^h, 2_\alpha^{h_2}, 3^{h_3}, ..., (n-1)^{h_{n-1}}),$$

$$A' = \sum_{h < h_1 + h_2} M_3((1^\beta)^{h_1}, 2^{h_2}, (-2_\alpha)^{-h}\}) \frac{1}{\langle 1, 2 \rangle [1, 2]} M_{n-1}(n_\beta^{h_n}, 2_\alpha^h, 3^{h_3}, ..., (n-1)^{h_n}(3.12)$$

which are shown in second lines of Figure 1 and Figure 3, respectively. Here we have recovered helicities for all legs and taken into account the fact that terms with $h_n + h_1 - h < 0$ vanish in A and those with $h_1 + h_2 - h < 0$ vanish in A'.[6]

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[6]This is because if, say in A', $h_1+h_2-h < 0$, then the three particle amplitude must be holomorphic.

To proceed, first we notice from Eq.(3.12) that two $n-1$ particle amplitudes in A and A' are almost the same, except that the first two helicities, (h, h_2) in A and (h_n, h) in A' may not be the same. So we shall only keep these two variables in M_{n-1} for both A and A' below. For three particle amplitudes, one needs Eq.(3.1) and Eq.(3.2), where for illustration we shall keep $\kappa^{[-2]}$ and $\kappa'^{[-2]}$, although the four-particle test requires that both of them should vanish.

Now we are ready to discuss all possible helicity arrangements for particle 1, 2 and n. If $h_1 = h_n = -$, A vanishes since there is no term with $h_n + h_1 - h > 0$, so does A' for $h_1 = h_2 = -$. Therefore, given $B = B'$, we conclude that in the case $(h_n, h_1, h_2) = (-, -, -)$, such two good deformations on $(1, 2)$ and $(1, n)$ produce the same result.

If $h_1 = -$ and $h_n = +$, then we must have $h = -$ in A. So we can obtain A by Eq.(3.2) as

$$A = \kappa'^{[0]} \frac{\langle 1, 2 \rangle^3}{\langle n, 2 \rangle^3} \langle n, 1 \rangle^{-1} M_{n-1}(-, h_2). \tag{3.13}$$

Likewise, if $h_1 = -$ and $h_2 = +$, then we have $h = -$ in A', which gives

$$A' = \kappa'^{[0]} \frac{\langle n, 1 \rangle^3}{\langle n, 2 \rangle^3} \langle 1, 2 \rangle^{-1} M_{n-1}(h_n, -). \tag{3.14}$$

Given $B = B'$, for $(h_n, h_1, h_2) = (-, -, +)$, the bad deformation on $(1, 2)$ with $(-, +)$ gives vanishing A, which is different from generically non-vanishing A' given by good deformation on $(1, n)$ with $(-, -)$. Similarly, for $(h_n, h_1, h_2) = (+, -, -)$, the bad deformation on $(1, n)$ with $(-, +)$ gives a different answer from that obtained by the good deformation on $(1, 2)$ with $(-, -)$. Finally, for $(h_n, h_1, h_2) = (+, -, +)$, the two deformations on $(1, 2)$ and $(1, n)$ are both bad ones with helicities $(-, +)$, which give different answers from each other generically. Therefore, as promised, using purely S-matrix arguments, we have also derived that the bad deformation with helicities $(-, +)$ can not be used in BCFW construction.

For $h_1 = +$, the deformations on $(1, 2)$ and $(1, n)$ are always good. After some algebraic calculations, the corresponding result can be obtained as

$$A = \kappa'^{[0]} \frac{\langle n, 2 \rangle}{\langle 1, 2 \rangle} \langle n, 1 \rangle^{-1} M_{n-1}(h_n, h_2) + \delta_{h_n, +} \kappa'^{[-2]} \frac{\langle 1, 2 \rangle}{\langle n, 2 \rangle} [n, 1]^2 \langle n, 1 \rangle^{-1} M_{n-1}(-, h_2), \tag{3.15}$$

and

$$A' = \kappa'^{[0]} \frac{\langle n, 2 \rangle}{\langle n, 1 \rangle} \langle 1, 2 \rangle^{-1} M_{n-1}(h_n, h_2) + \delta_{h_2, +} \kappa'^{[-2]} \frac{\langle n, 1 \rangle}{\langle n, 2 \rangle} [1, 2]^2 \langle 1, 2 \rangle^{-1} M_{n-1}(h_n, -), \tag{3.16}$$

However, both λ^{1^β} and $\lambda^{2\alpha}$ are proportional to λ^2, thus the amplitude possesses a factor $\langle 2, 2 \rangle^{-h_1 - h_2 + h}$, which vanishes.

where we can immediately recognize that the terms with $\kappa'^{[0]}$ are equal to each other while those with $\kappa'^{[-2]}$ are generally not. Therefore, given $B = B'$, if $(h_n, h_1, h_2) = (-, +, -)$, there are only first terms in both A and A', so we get the same result for such two good deformations. For $(h_n, h_1, h_2) = (-, +, +)$, $(+, +, -)$, or $(+, +, +)$, the two good deformations yield the same answer if and only if $\kappa'^{[-2]} = 0$.

Therefore, given $\kappa'^{[-2]} = 0$, which has been guaranteed by the four-particle test, we conclude that, as long as no bad deformation is involved,

$$M_n^{(1^\alpha, 2_\alpha)} = M_n^{(1^\beta, n_\beta)}. \tag{3.17}$$

Now consider $M_n^{(2^{\alpha'}, 1_{\alpha'})} = M_n^{(n^{\beta'}, 1_{\beta'})}$. If all deformations are good, the proof of this equality goes exactly the same way as the proof of Eq.(3.17), only with all helicities flipped and $\lambda \leftrightarrow \tilde{\lambda}$.

Finally, since any partial amplitude is cyclic symmetric, our proof can be applied to any leg i of an n particle partial amplitude, $1 \le i \le n$, as long as any deformation involved is good.

To summarize, we have proved that $M_n^{(i^\mu, (i-1)_\mu)} = M_n^{(i^\nu, (i+1)_\nu)}$ and $M_n^{((i-1)^{\mu'}, i_{\mu'})} = M_n^{((i+1)^{\nu'}, i_{\nu'})}$ hold if and only if the deformation involved is good one.

3.4 Proof of the correct factorizations of amplitudes

The final step is to show that the amplitude constructed by deforming adjacent particles is the correct physical amplitude. As discussed before, it is sufficient to check if the amplitude has all the correct factorization channels, which for partial amplitudes are only made up of adjacent momenta.

The statement that the amplitude has correct factorizations means if we send the momentum of a channel on-shell, the amplitude should contain a singular term which is the product of two sub-amplitudes with the propagator of this channel, plus other non-singular terms. We have supposed this is true for M_k with $3 \le k \le n-1$, and now we prove that the n particle partial amplitude constructed by recursion relations also have correct factorizations for any channel being sent on-shell.

Suppose we obtain the amplitude by deforming $(1, n)$, which is given by Eq.(3.9). Then any propagator appearing in Eq.(3.9) comes from the channel in the form $s_{j,...,k}$ with $1 \le j < k \le n-2$ or $3 \le j < k \le n$. We now want to check that, if one sends the momentum of such a channel on-shell, i.e., $s_{j,...,k} \to 0$, the amplitude really becomes a product of two sub-amplitudes, with the singular propagator of this channel, plus other non-singular terms.

First, we know that sub-amplitudes have the correct factorization when $s_{j,...,k} \to 0$, i.e.,

$$M_{i+1}(1^\beta, ..., j, ..., k, ..., i, -(1^\beta \oplus ... \oplus i)) = M_{k-j+2}(j, ..., k, -(j \oplus ... \oplus k))$$
$$\times \frac{1}{|j \oplus ... \oplus k|^2} M_{i-k+j+1}(1^\beta, ..., j \oplus ... \oplus k, ..., i, -(1^\beta \oplus ... \oplus i))$$
$$+ \textit{non-singular terms} \tag{3.18}$$

for $1 \le j < k \le i \le n - 2$, and

$$M_{n-i+1}(i+1, ..., j, ..., k, ..., n_\beta, -((i+1) \oplus ... \oplus n_\beta)) = M_{k-j+2}(j, ..., k, -(j \oplus ... \oplus k))$$
$$\times \frac{1}{|j \oplus ... \oplus k|^2} M_{n-i-k+j+1}(i+1, ..., j \oplus ... \oplus k, ..., n_\beta, (i+1) \oplus ... \oplus n_\beta)$$
$$+ \textit{non-singular terms} \tag{3.19}$$

for $2 \le i < j < k \le n$.

Then by Eq.(3.9), Eq.(3.18), and Eq.(3.19), we obtain

$$M_n^{(1^\beta, n_\beta)} = [\sum_{i=k}^{n-2} M_{n-i+1}(i+1, ..., n_\beta, -((i+1) \oplus ... \oplus n_\beta)) \frac{1}{|(i+1) \oplus ... \oplus n|^2}$$
$$\times M_{i-k+j+1}(1^\beta, ..., j \oplus ... \oplus k, ..., i, (i+1) \oplus ... \oplus n_\beta)$$
$$+ \sum_{i=2}^{j-1} M_{i+1}(1^\beta, ..., i, -(1^\beta \oplus ... \oplus i)) \frac{1}{|1 \oplus ... \oplus i|^2}$$
$$\times M_{n-i-k+j+1}(i+1, ..., j \oplus ... \oplus k, ..., n_\beta, 1^\beta \oplus ... \oplus i)]$$
$$\times \frac{1}{|j \oplus ... \oplus k|^2} M_{k-j+2}(j, ..., k, -(j \oplus ... \oplus k)) + \textit{non-singular terms} \tag{3.20}$$

for $1 \le j < k \le i \le n - 2$, or $2 \le i < j < k \le n$. Note that all the terms in [ ], i.e., those from the first to the fourth line, are the factorizations by deforming $(1, n)$ of an $n - k + j$ particle amplitude, which are obtained by replacing all the legs from j to k with a single on-shell leg $j \oplus ... \oplus k$, thus we have

$$M_n^{(1^\beta, n_\beta)} = M_{n-j+k}(1, ..., j-1, j \oplus ... \oplus k, k+1, ..., n)$$
$$\times \frac{1}{|j \oplus ... \oplus k|^2} M_{k-j+2}(j, ..., k, -(j \oplus ... \oplus k)) + \textit{non-singular terms}, \tag{3.21}$$

which means the n particle partial amplitude also has the correct factorization when $s_{j,...,k} \to 0$ for $1 \le j < k \le i \le n - 2$, or $2 \le i < j < k \le n$. Notice that $s_{j,...,k} = s_{k+1,...,n,1,...,j-1}$, so the correct factorization channels include all possible channels of the partial amplitude except $s_{n,1}$.

However, an important thing we need to check is the inclusion of both holomorphic and anti-holomorphic collinear limits. Since $p_{i,i+1}^2 = \langle i, i+1 \rangle [i, i+1]$, we need to take care of two separate cases[7], i.e., the holomorphic pole, $\langle i, i+1 \rangle \to 0$ while $[i, i+1] \neq 0$, and anti-holomorphic pole, $[i, i+1] \to 0$ while $\langle i, i+1 \rangle \neq 0$. From Eq.(3.9), we can see that $M_n^{(1^\beta, n_\beta)}$ has the correct factorizations at the anti-holomorphic pole from the channel $s_{1,2}$, the holomorphic pole from channel $s_{n-1,n}$, and at both holomorphic and anti-holomorphic poles from all other channels except $s_{n,1}$.

In other words, we have not shown that $M_n^{(1^\beta, n_\beta)}$ given by Eq.(3.9) has the correct factorizations at the anti-holomorphic pole from $s_{n-1,n}$, the holomorphic pole from $s_{1,2}$, and both poles from the channel $s_{n,1}$. Nevertheless, amplitudes constructed by different deformations, such as $M_n^{(1^\alpha, 2_\alpha)}$ given by Eq.(3.8) can have correct factorizations at (some of) these poles, then since we have equalities relating them, they give the same amplitude as a rational function of external momenta, which implies that $M_n^{(1^\beta, n_\beta)}$ must also have the correct factorizations at these poles.

If there are still some poles that are not explicitly included in either Eq.(3.9) or Eq.(3.8), then more deformations which give the same function are needed. Therefore, our strategy below is to find a chain of equalities which relates different deformations to ensure each of them has the correct factorizations at all poles, including holomorphic and anti-holomorphic collinear limits.

Let us show the correct factorizations of the n particle rational function constructed by BCFW recursion relations for all helicity configurations. First we discuss the case with $h_1 = +$, then there are four possibilities for (h_n, h_1, h_2), which are $(-, +, -), (+, +, +), (-, +, +)$ and $(+, +, -)$.

If $(h_n, h_1, h_2) = (-, +, -)$, then our proof in 3.3 gives $M_n^{(1^\alpha, 2_\alpha)} = M_n^{(1^\beta, n_\beta)}$. But we can also move a step towards the particle $n-1$, i.e., for any h_{n-1}, we always have $M_n^{((n-1)^\mu, n_\mu)} = M_n^{(1^\beta, n_\beta)}$ since $h_n = -$. Similarly, for any h_3, we have $M_n^{(3^\nu, 2_\nu)} = M_n^{(1^\alpha, 2_\alpha)}$ since $h_2 = -$. We thus conclude

$$M_n^{((n-1)^\mu, n_\mu)} = M_n^{(3^\nu, 2_\nu)}. \tag{3.22}$$

Now we can check their factorizations at various poles. We have shown that $M_n^{((n-1)^\mu, n_\mu)}$ has correct factorizations at all possible poles except the holomorphic pole

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[7]An example for illustration is a five particle amplitude with certain helicity configuration, i.e., $M_5 \propto \frac{\langle 1,2 \rangle [3,4]}{[1,2] \langle 3,4 \rangle}$, where it is easy to see that for real collinear limits, i.e., as both $\langle 1, 2 \rangle$ and $[1, 2]$ go to zero, this function is not singular. In fact there is no real collinear limit or factorization that can detect this. However, an anti-holomorphic factorization limit, i.e., $[1, 2] \to 0$ while $\langle 1, 2 \rangle \neq 0$, detects it and the inclusion of this anti-holomorphic collinear limit is needed for the function to be the correct physical amplitude.

from $s_{n-2,n-1}$, the anti-holomorphic pole from $s_{n,1}$, and both poles from $s_{n-1,n}$. For $M_n^{(3^\nu, 2_\nu)}$, we have shown it has the correct factorizations at all possible poles except the holomorphic pole from $s_{3,4}$, anti-holomorphic pole from $s_{1,2}$ and both poles from $s_{2,3}$.

Since they are the same function, $M_n^{((n-1)^\mu, n_\mu)}$ must have the same factorizations as $M_n^{((3)^\nu, 2_\nu)}$, and vise versa. The conclusion is that each of them does contain all possible poles of an n particle partial amplitude and has the correct factorizations at each of them, which means that either by $M_n^{((n-1)^\mu, n_\mu)}$ or $M_n^{(3^\nu, 2_\nu)}$, or the same function given by other deformations, we have obtained the correct amplitude.

If $(h_n, h_1, h_2) = (+, +, +)$, then we have $M_n^{(2^{\alpha'}, 1_{\alpha'})} = M_n^{(n^{\beta'}, 1_{\beta'})}$, $M_n^{(n^{\mu'}, (n-1)_{\mu'})} = M_n^{(n^{\beta'}, 1_{\beta'})}$, and $M_n^{(2^{\nu'}, 3_{\nu'})} = M_n^{(2^{\alpha'}, 1_{\alpha'})}$. Therefore, we arrive at

$$M_n^{(n^{\mu'}, (n-1)_{\mu'})} = M_n^{(2^{\nu'}, 3_{\nu'})}, \tag{3.23}$$

where the L.H.S. explicitly has the correct factorizations at all possible poles except the holomorphic pole from $s_{n,1}$, anti-holomorphic pole from $s_{n-2,n-1}$ and both poles from $s_{n-1,n}$, so does the R.H.S. at all possible poles except the holomorphic pole from $s_{1,2}$, anti-holomorphic pole from $s_{3,4}$, and both poles from $s_{2,3}$. Just as in the case $(h_n, h_1, h_2) = (-, +, -)$, both of them have correctly included all possible poles and yield the correct partial amplitude.

For $(h_n, h_1, h_2) = (-, +, +)$, we can only get a shorter chain of equalities, i.e.,

$$M_n^{(1^\alpha, 2_\alpha)} = M_n^{(1^\beta, n_\beta)} = M_n^{((n-1)^\mu, n_\mu)}, \tag{3.24}$$

for any h_{n-1}. This is because if we want to extend it to the deformation on $(3, 2)$, we may encounter the bad deformation if $h_3 = -$. However, we now show this shorter chain is enough for our purpose.

We have shown that $M_n^{(1^\alpha, 2_\alpha)}$ explicitly has the correct factorizations at all possible poles except the holomorphic pole from $s_{n,1}$, anti-holomorphic pole from $s_{2,3}$, and both poles from $s_{1,2}$, so does $M_n^{(1^\beta, n_\beta)}$ at all possible poles except the holomorphic pole from $s_{1,2}$, anti-holomorphic pole from $s_{n-1,n}$, and both poles from $s_{n,1}$.

Since they are the same rational function, both $M_n^{(1^\alpha, 2_\alpha)}$ and $M_n^{(1^\beta, n_\beta)}$ have the correct factorizations at all poles except the holomorphic poles from $s_{n,1}$ and $s_{1,2}$.

Now since they are also the same function as $M_n^{((n-1)^\mu, n_\mu)}$ which has the correct factorizations at both poles from $s_{1,2}$ and the holomorphic pole from $s_{n,1}$, we can see that all factorization channels of a partial amplitude are correctly included and any of these deformations has given the correct answer. The case with $(h_n, h_1, h_2) = (+, +, -)$ can also be similarly proved by the chain $M_n^{(1^\beta, n_\beta)} = M_n^{(1^\alpha, 2_\alpha)} = M_n^{(3^\nu, 2_\nu)}$.

All these discussions can apply to $h_1 = -$, only with all helicities flipped and $\lambda \leftrightarrow \tilde{\lambda}$. To summarize, we have proved the weak version for n particle partial amplitude, namely any n particle partial amplitude can be consistently constructed from lower amplitudes by the deformation on any pair of adjacent particles as long as it is a good deformation, and the resultant function possesses all the correct factorization channels.

By induction, the conclusion is that for spin 1 massless particles in four dimensional Minkowski spacetime, given Poincare symmetry, any tree-level amplitude can be constructed consistently from lower amplitudes and eventually from basic three particle amplitudes via BCFW recursion relations, if and only if, (1). there is no coupling constants with negative dimensions, i.e., $\kappa^{[-2]} = \kappa'^{[-2]} = 0$; and (2). dimensionless coupling constants must conform to Jacobi condition, i.e., $\sum_e (f_{ade} f_{ebc} + f_{ace} f_{edb} + f_{abe} f_{ecd}) = 0$.

4. Conclusion and Discussions

In this paper we have investigated the consistency conditions on scattering amplitudes of spin 1 massless particles purely from the S-matrix arguments. Instead of using Yang-Mills Lagrangian and its Feynman diagrams, we directly constructed tree-level amplitudes from lower amplitudes by BCFW recursion relations and proved this can be consistently done and the resultant functions are indeed the correct physical amplitudes. The main conclusions of this paper and [1] can be summarized as follows:
(1). Candidates for n particle amplitudes are constructed from lower amplitudes by BCFW recursion relations using a pair of deformed particles with complex momenta.
(2). Three particle amplitudes are non-perturbatively determined by Poincare symmetry and the tree-level four-particle test requires the absence of negative dimension couplings and dimensionless coupling constants to be the structure constants of a Lie group.
(3). Equalities relating candidates for the tree-level n particle amplitudes are obtained and are shown to have correct factorizations at all possible poles, including holomorphic and anti-holomorphic collinear limits, which ensure them to be the correct physical amplitudes.

A remark on the strong version of consistency conditions is necessary. We have only proved such equalities as $M_n^{(i,i-1)} = M_n^{(i,i+1)}$ and $M_n^{(i-1,i)} = M_n^{(i+1,i)}$, which are enough to ensure any of these deformations, as long as it is a good one, yields the correct physical amplitude. This in turn gives us a single stronger chain of equalities,

$$M_n^{(i,i+1)} = M_n^{(j,j+1)} = M_n^{(i,i-1)} = M_n^{(j,j-1)}, \tag{4.1}$$

for any $1 \le i, j \le n$ as long as the concerned deformations are all good.

What has not been proved is whether this equals any other good deformation on non-adjacent particles, and a direct comparison of amplitudes constructed by deforming non-adjacent particles with those constructed by deforming adjacent particles, is at least not very straightforward, due to the explicit use of color-decomposition.

A strategy to prove the strong version is to use the fact that by any good deformation on adjacent particles we have obtain the correct physical amplitude M_n, which in turn can be deformed on any pair of non-adjacent particles, say (l, m), as long as the deformation is good. The key requirement for M_n to be constructed from lower amplitudes by deforming (l, m) is

$$\lim_{z \to \infty} M_n^{l,m}(z) = 0, \tag{4.2}$$

which ensures $M_n^{(l,m)} \equiv M_n^{(l,m)}(0)$ to be expressed as the sum of residues at finite poles and the result is exactly BCFW construction.

Let us assume the strong version of consistency conditions on lower amplitudes, which means any lower amplitude vanishes when a pair of particles are deformed with parameter z going to infinity. Now since n particle amplitude has been proven to be given by $M_n^{(i,i+1)}$, if we deform a pair of particles (l, m) of it and send the parameter z to infinity, it should be possible to show that every term appearing in $M_n^{(i,i+1)}$ vanishes because lower amplitudes vanish in this limit, from which the strong version of consistency conditions can follow by induction.

There are several future directions worthy of investigations. An obvious one is to generalize the proof of consistency conditions to theories of particles with other spins, such as theories of spin 2 massless particles and theories of particles with lower spins coupled to spin 1 or spin 2 particles. Although the lack of color-decomposition makes such generalizations apparently difficult, it has been pointed out in [8, 10] that theories with spin 2 particles, such as General Relativity and Supergravity, which do not possess color-decomposition, have simpler structure in their amplitudes due to even better vanishing behaviors at infinite momenta, thus a similar proof of consistency conditions on amplitudes in these gravitational theories is highly desirable. Supersymmetric theories are notable here since supersymmetry can relate amplitudes of particles with lower spins to the better behaved amplitudes of highest spin particle, as have been used in supersymmetric extension of BCFW in [10]. It will be intriguing to see this purely from the S-matrix arguments.

In addition, it is interesting to see if this proof can be generalized to other spacetime dimensions. Since BCFW recursion relations have been proved for $D \geq 4$ dimensional Yang-Mills theory and perturbative gravity from Lagrangian point of view [8], there

should be a direct generalization of our proof to higher dimensions although the convenient spinor techniques may not be used in this case. A crucial insight is the enhanced Lorentz symmetry of effective theory at large complex momenta and it is desirable to uncover it without Lagrangian or Feynman diagrams. On the other hand, consistency conditions on theories in three dimensions are also interesting since exactly solvable models are available there.

A direction of more significance is the investigation of purely S-matrix argument for simplicities of loop-level amplitudes and their consistency conditions. From quantum field theory point of view, tree-level consistent theories can be anomalous at loop-levels, thus it is important to extend our analysis to the loop-levels. On the other hand, remarkable simplicities in loop-level amplitudes which are not manifest from local quantum field theory and its Feynman diagrams, especially for maximal supersymmetric theories in four dimensional spacetime, i.e., $\mathcal{N} = 4$ SYM theory and $\mathcal{N} = 8$ Supergravity, imply that a purely S-matrix understanding is necessary and desirable. For example, it will be interesting to derive the absence of triangles, bubbles and rational terms at one-loop level for maximal supersymmetric theories from a purely S-matrix argument.

An even more ambitious possibility, as emphasized in [10], is to search for a dual formulation of local quantum field theory and its Feynman diagrams which manifests BCFW construction as well as loop-level simplicities of amplitudes. Since now a purely S-matrix argument for consistency conditions on amplitudes is available, the existence of such a dual formulation has been put onto a more solid ground and the proof here can be considered as a starting point for the construction of the dual theory.

Acknowledgement

Appendix: Expressions of deformed momenta

Here we shall work out the specific expressions for $1^\alpha, 2_\alpha, 1^\beta, n_\beta, 1^\beta \oplus 2$ and $n \oplus 1^\alpha$. First 1^α and 2_α come from the deformation $(1^\alpha, 2_\alpha)$, i.e.,

$$
\begin{aligned}
\lambda^{(1^\alpha)} &= \lambda^{(1)} + z(\alpha)\lambda^{(2)}, \tilde{\lambda}^{(1^\alpha)} = \tilde{\lambda}^{(1)}, \\
\lambda^{(2_\alpha)} &= \lambda^{(2)}, \tilde{\lambda}^{(2_\alpha)} = \tilde{\lambda}^{(2)} - z(\alpha)\tilde{\lambda}^{(1)},
\end{aligned}
\tag{1}
$$

whereby the parameter $z(\alpha)$ can be obtained as $z(\alpha) = -\langle n, 1 \rangle / \langle n, 2 \rangle$ by leaving on-shell the momentum of $n \oplus 1^\alpha$,

$$
P = \lambda^{(1^\alpha)}\tilde{\lambda}^{(1)} + \lambda^{(n)}\tilde{\lambda}^{(n)}.
\tag{2}
$$

Furthermore taking the inner product of the momentum with the $\lambda^{(n)}$ yields zero by using $\langle n, 1^\alpha \rangle = 0$. Whence we know the left-handed part can always be set equal to $\lambda^{(n)}$ due to the little group transformation. Thus by taking the inner product of the momentum with $\lambda^{(1)}$, we obtain the final expression of $n \oplus 1^\alpha$ as

$$
\lambda^{(n \oplus 1^\alpha)} = \lambda^{(n)}, \tilde{\lambda}^{(n \oplus 1^\alpha)} = \frac{\langle 1, 2 \rangle}{\langle n, 2 \rangle}\tilde{\lambda}^{(1)} + \tilde{\lambda}^{(n)}.
\tag{3}
$$

Similarly, 1^β and n_β come from the deformation $(1^\beta, n_\beta)$, i.e.,

$$
\begin{aligned}
\lambda^{(1^\beta)} &= \lambda^{(1)} + z(\beta)\lambda^{(n)}, \tilde{\lambda}^{(1^\beta)} = \tilde{\lambda}^{(1)}, \\
\lambda^{(n_\beta)} &= \lambda^{(n)}, \tilde{\lambda}^{(n_\beta)} = \tilde{\lambda}^{(n)} - z(\beta)\tilde{\lambda}^{(1)},
\end{aligned}
\tag{4}
$$

whereby the parameter $z(\beta)$ can be obtained as $z(\beta) = -\langle 1, 2 \rangle / \langle n, 2 \rangle$ by leaving on-shell the momentum of $1^\beta \oplus 2$,

$$
P = \lambda^{(1^\beta)}\tilde{\lambda}^{(1)} + \lambda^{(2)}\tilde{\lambda}^{(2)}.
\tag{5}
$$

Furthermore taking the inner product of the momentum with the $\lambda^{(2)}$ yields zero by using $\langle 1^\beta, 2 \rangle = 0$. Whence we know the left-handed part can always be set equal to $\lambda^{(2)}$ due to the little group transformation. Thus by taking the inner product of the momentum with $\lambda^{(1)}$, we obtain the final expression of $1^\beta \oplus 2$ as

$$
\lambda^{(1^\beta \oplus 2)} = \lambda^{(2)}, \tilde{\lambda}^{(1^\beta \oplus 2)} = \frac{\langle n, 1 \rangle}{\langle n, 2 \rangle}\tilde{\lambda}^{(1)} + \tilde{\lambda}^{(2)}.
\tag{6}
$$

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